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String scattering off the (2+1)-dimensional rotating black hole

Makoto Natsuume,^{*} Norisuke Sakai,[†] and Masamichi Sato [‡]

*Department of Physics
Tokyo Institute of Technology
Oh-okayama, Meguro, Tokyo 152, Japan*

ABSTRACT

The $SL(2, R)/Z$ WZW orbifold describes the (2+1)-dimensional black hole which approaches anti-de Sitter space asymptotically. We study the $1 \rightarrow 1$ tachyon scattering off the rotating black hole background and calculate the Hawking temperature using the Bogoliubov transformation.

^{*}makoto@th.phys.titech.ac.jp, JSPS Research Fellow

[†]nsakai@th.phys.titech.ac.jp

[‡]sato@th.phys.titech.ac.jp

1 Introduction

The Wess-Zumino-Witten (WZW) model is a useful framework to study string theory in curved spacetime. One simple example is the $SL(2, R)/U(1)$ WZW coset which is known as Witten Black Hole [1]. Another simple example is the $SL(2, R)/\mathbb{Z}$ WZW orbifold, which describes a three-dimensional black hole in asymptotically anti-de Sitter space [2, 3]. The black hole was originally found as a solution to general relativity by Bañados, Teitelboim, and Zanelli (BTZ) [4], but it was quickly realized that a slight modification of the solution yields a solution to the bosonic string theory.

The purpose of this note is to study the $1 \rightarrow 1$ string scattering in the rotating BTZ black hole background. We solve the tachyon equation in the linearized approximation and derive the reflection coefficient for the scattering. We also derive the Hawking temperature using the Bogoliubov transformation.

String scatterings in various geometries have been studied in refs. [5, 6, 7]. In particular, ref. [7] studies the string scattering in the static ($J = 0$) BTZ black hole background. In the context of the BTZ black hole in general relativity, the scattering of a massless conformally coupled scalar field has been studied in ref. [8].

2 Review

We will briefly review the BTZ black hole; for more details, see ref. [9]. The simplest solution of the BTZ black hole is the $(M, J) = (1, 0)$ case. It is given by

$$\begin{aligned} ds^2 &= 2(k-2) \left\{ -(\hat{r}^2 - 1)dt^2 + \frac{d\hat{r}^2}{\hat{r}^2 - 1} + \hat{r}^2 d\hat{\varphi}^2 \right\}, \\ \Phi &= \text{const.}, \\ H_{\mu\nu\rho} &= -2(k-2)\epsilon_{\mu\nu\rho}. \end{aligned} \tag{1}$$

We set $\alpha' = 2$. Here, we identify $\hat{\varphi} \approx \varphi + 2\pi$; this corresponds to a choice of the orbifolding. Starting from a level k $SL(2, R)$ WZW model, it can

be shown that the above solution is an exact solution to the bosonic string theory. The normalization is obtained as follows. k appears since we consider the level k WZW model; -2 is the well-known shift of k by the amount of c_v [10], where c_v is the quadratic Casimir of the adjoint representation of the group. The three-form field strength $H_{\mu\nu\rho}$ is necessary by the Wess-Zumino term and plays the role of negative cosmological constant. So, the solution approaches anti-de Sitter space asymptotically.

Since the central charge of the WZW model is given by $c = 3k/(k - 2)$, $k = 52/23$ in order to get $c = 26$. In the discussion of the BTZ black hole, a dimensionful parameter l is often used. This parameter can be introduced by scaling \hat{t} and \hat{r} . The relation of l , the cosmological constant Λ , and the level k is given by $l^2 = -\Lambda^{-1} = 2(k - 2)$.

An important property of the BTZ black hole is that the general solution is obtained simply by a different choice of the orbifolding. This property will be essential in solving the tachyon equation. First, make the transformation

$$\hat{t} = r_+ t - r_- \varphi, \quad \hat{\varphi} = r_+ \varphi - r_- t, \quad \hat{r}^2 = \frac{r^2 - r_-^2}{r_+^2 - r_-^2}. \quad (2)$$

The reason of making this transformation is because we will make the identification in terms of φ rather than $\hat{\varphi}$. By the transformation, the metric (1) becomes

$$ds^2 = 2(k - 2) \left\{ -(r^2 - M)dt^2 - Jdtd\varphi + r^2 d\varphi^2 + \left(r^2 - M + \frac{J^2}{4r^2}\right)^{-1} dr^2 \right\}. \quad (3)$$

This time we identify $\varphi \approx \varphi + 2\pi$ as the orbifolding. Here, $M = r_+^2 + r_-^2$; $J = 2r_+ r_-$; and r_{\pm} are the inner and outer horizons of the black hole.

3 $1 \rightarrow 1$ Scattering

Consider the effective action for the tachyon T . The spacetime action is

$$S(T) = \int d^3 X \sqrt{-G} e^{-2\Phi} (G^{\mu\nu} \partial_\mu T \partial_\nu T + m^2 T^2 + aT^3 + \dots), \quad (4)$$

where $m^2 = -2$. We expand the tachyon field equation in powers of the ingoing tachyon. To first order in the tachyon, the field equation is

$$-\frac{1}{e^{-2\Phi}\sqrt{-G}}\partial_\mu G^{\mu\nu}e^{-2\Phi}\sqrt{-G}\partial_\nu T + m^2 T = 0. \quad (5)$$

Our task is to solve eq. (5) in the background (3). Although $m^2 = -2$, we will parametrize $m^2 = -(4\lambda^2 + 1)/2(k - 2)$. Expressions become simpler in this parametrization. From the actual value of m^2 , $\lambda^2 = 1/92$. The parametrization appears naturally in the representation theory of the $SL(2, R)$ affine Kac-Moody algebra; the tachyon belongs to the continuous series representation of the global $SL(2, R)$.

Substituting the metric (3) into (5), we get

$$\begin{aligned} (r^2 - M + \frac{J^2}{4r^2})\frac{\partial^2 T}{\partial r^2} + (3r - \frac{M}{r} - \frac{J^2}{4r^3})\frac{\partial T}{\partial r} \\ - (r^2 - M + \frac{J^2}{4r^2})^{-1}\left\{\frac{\partial^2}{\partial t^2} + \frac{M - r^2}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{J}{r^2}\frac{\partial^2}{\partial \varphi \partial t}\right\}T + (4\lambda^2 + 1)T = 0. \end{aligned} \quad (6)$$

To solve this equation, expand the field in terms of modes:

$$T(r, t, \varphi) = \sum_N \int dE T_{EN}(r) e^{-iEt} e^{-iN\varphi}. \quad (7)$$

Here, $N \in \mathbb{Z}$. Eq. (6) is most easily solved by first making the change of coordinates (2). Then the sum (7) is rearranged as

$$T(r, \hat{t}, \hat{\varphi}) = \mathbf{S}_{\hat{N}} \mathbf{S}_{\hat{E}} T_{\hat{E}\hat{N}}(r) e^{-i\hat{E}\hat{t}} e^{-i\hat{N}\hat{\varphi}}. \quad (8)$$

Here, \mathbf{S} denotes the summation over the modes \hat{E} and \hat{N} . Also, we set $E = r_+ \hat{E} - r_- \hat{N}$ and $N = -r_- \hat{E} + r_+ \hat{N}$. By the coordinate transformation and the mode expansion, eq. (6) becomes

$$(\hat{r}^2 - 1)\frac{d^2 T_{\hat{E}\hat{N}}}{d\hat{r}^2} + (3\hat{r} - \frac{1}{\hat{r}})\frac{dT_{\hat{E}\hat{N}}}{d\hat{r}} + (\frac{\hat{E}^2}{\hat{r}^2 - 1} - \frac{\hat{N}^2}{\hat{r}^2} + 4\lambda^2 + 1)T_{\hat{E}\hat{N}} = 0. \quad (9)$$

By changing variables to $z = 1 - \hat{r}^2 = \frac{r_+^2 - r_-^2}{r_+^2 - r_-^2}$ and

$$T_{\hat{E}\hat{N}}(z) = z^{i\hat{E}/2} (1 - z)^{i\hat{N}/2} \Psi_{\hat{E}\hat{N}}(z), \quad (10)$$

we get the standard hypergeometric differential equation for $\Psi_{\hat{E}\hat{N}}(z)$:

$$z(1-z)\frac{d^2\Psi_{\hat{E}\hat{N}}}{dz^2} + \{c - (a+b+1)z\}\frac{d\Psi_{\hat{E}\hat{N}}}{dz} - ab\Psi_{\hat{E}\hat{N}} = 0, \quad (11)$$

where

$$\begin{aligned} a &= 1/2 + i\lambda + i(\hat{E} + \hat{N})/2, \\ b &= 1/2 - i\lambda + i(\hat{E} + \hat{N})/2, \\ c &= 1 + i\hat{E}. \end{aligned} \quad (12)$$

The general solution is written in terms of Kummer's fundamental system of solutions for the hypergeometric differential equation:

$$T_{\hat{E}\hat{N}}(z) = c_1 U_{\hat{E}\hat{N}} + c_2 V_{\hat{E}\hat{N}} \quad \text{for } |z| < 1, \quad (13)$$

$$= \bar{c}_1 \bar{U}_{\hat{E}\hat{N}} + \bar{c}_2 \bar{V}_{\hat{E}\hat{N}} \quad \text{for } |z| > 1, \quad (14)$$

where

$$U_{\hat{E}\hat{N}} = z^{i\hat{E}/2}(1-z)^{i\hat{N}/2}F(a, b, c; z), \quad (15)$$

$$V_{\hat{E}\hat{N}} = z^{i\hat{E}/2}(1-z)^{i\hat{N}/2}z^{1-c}F(a-c+1, b-c+1, 2-c; z), \quad (16)$$

$$\bar{U}_{\hat{E}\hat{N}} = z^{i\hat{E}/2}(1-z)^{i\hat{N}/2}(-z)^{-b}F(b, b-c+1, b-a+1; 1/z), \quad (17)$$

$$\bar{V}_{\hat{E}\hat{N}} = z^{i\hat{E}/2}(1-z)^{i\hat{N}/2}(-z)^{-a}F(a, a-c+1, a-b+1; 1/z). \quad (18)$$

The modes (U, V) are analogous to the Rindler modes. These modes obey $V_{\hat{E}\hat{N}}^\lambda = (U_{\hat{E}-\hat{N}}^{-\lambda})^*$ and $\bar{V}_{\hat{E}\hat{N}}^\lambda = (\bar{U}_{-\hat{E}-\hat{N}}^\lambda)^*$. The relation between \bar{c}_i and c_i are obtained using the linear transformation properties of hypergeometric functions [11]:

$$\bar{c}_1 = c_1 \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} + c_2 \frac{\Gamma(2-c)\Gamma(a-b)}{\Gamma(a-c+1)\Gamma(1-b)}, \quad (19)$$

$$\bar{c}_2 = c_1 \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} + c_2 \frac{\Gamma(2-c)\Gamma(b-a)}{\Gamma(b-c+1)\Gamma(1-a)}. \quad (20)$$

The constants c_i are determined by imposing the appropriate boundary conditions below.

Near the horizon ($r \rightarrow r_+$ or $z \rightarrow 0$), the general solution approaches to

$$\begin{aligned} T_{\hat{E}\hat{N}}(z \rightarrow 0) &\simeq c_1(-z)^{i\hat{E}/2} + c_2(-z)^{-i\hat{E}/2} \\ &\simeq c_1 e^{i\hat{E} \ln \sqrt{r^2 - r_+^2}} + c_2 e^{-i\hat{E} \ln \sqrt{r^2 - r_+^2}}. \end{aligned} \quad (21)$$

Similarly, in the asymptotic region ($r \rightarrow \infty$ or $z \rightarrow -\infty$),

$$T_{\hat{E}\hat{N}}(z \rightarrow -\infty) \simeq \bar{c}_1(-z)^{-1/2+i\lambda} + \bar{c}_2(-z)^{-1/2-i\lambda}. \quad (22)$$

Here, we omit various constants and phases which are irrelevant to later discussion. Henceforth, we set $E > 0$ and $\lambda > 0$. Then, the first and second terms in (21) and (22) represent outgoing and ingoing modes respectively in the s -wave sector. Note that λ , not E , plays the role of the “radial momentum” from the asymptotic behavior (22). This is because of the unusual asymptotic geometry; the geometry is anti-de Sitter rather than Minkowski.

First we investigate the tachyon scattering off the black hole. The appropriate choice for the constants is $c_1 = 0$. Asymptotically this solution has both ingoing and outgoing modes, but only ingoing modes exist near the horizon:

$$T_{\hat{E}\hat{N}}(r \rightarrow r_+) \simeq e^{-i\hat{E} \ln \sqrt{r^2 - r_+^2}}, \quad (23)$$

and

$$T_{\hat{E}\hat{N}}(r \rightarrow \infty) \simeq \bar{c}_1 e^{-(1-2i\lambda) \ln r} + \bar{c}_2 e^{-(1+2i\lambda) \ln r}. \quad (24)$$

Here, c_2 has been normalized to unity. The reflection coefficient is now easily read as

$$\begin{aligned} R &= \left| \frac{\Gamma(a-b)\Gamma(b-c+1)\Gamma(1-a)}{\Gamma(a-c+1)\Gamma(1-b)\Gamma(b-a)} \right|^2 \\ &= \frac{\cosh \pi \left(\lambda - \frac{E+N}{2(r_+ - r_-)} \right) \cosh \pi \left(\lambda - \frac{E-N}{2(r_+ + r_-)} \right)}{\cosh \pi \left(\lambda + \frac{E+N}{2(r_+ - r_-)} \right) \cosh \pi \left(\lambda + \frac{E-N}{2(r_+ + r_-)} \right)}. \end{aligned} \quad (25)$$

In general, tachyon equations in rotating black hole backgrounds are more complicated than the ones in non-rotating black hole backgrounds and may

not be easy to solve. The BTZ black hole case is special, because the transformation (2) relates the $J \neq 0$ metric to the $J = 0$ metric.¹ This fact was crucial to solve the tachyon equation (6). In fact, the eq. (9) is nothing but the tachyon equation for the static $M = 1$ BTZ black hole. This equation appeared in the study of string scattering in the static BTZ black hole background [7].

This property, the $J \neq 0$ metric can be transformed to the $J = 0$ metric, is special to the BTZ solution. Most of the other rotating black hole metrics cannot be cast in the form of non-rotating black hole metrics. For example, written in Boyer-Lindquist coordinates, the Kerr solution has the form

$$ds^2 = -\frac{\Delta}{\rho^2}\{dt - a \sin^2 \theta d\varphi\}^2 + \frac{\sin^2 \theta}{\rho^2}\{(r^2 + a^2)d\varphi - a dt\}^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \quad (26)$$

where

$$\begin{aligned} \Delta &= r^2 - 2Mr + a^2, \\ \rho^2 &= r^2 + a^2 \cos^2 \theta, \end{aligned} \quad (27)$$

and a is the angular momentum per unit mass. This metric cannot be written in the form of the Schwarzschild solution because the expressions inside curly brackets in (26) are not integrable.

4 Hawking Radiation

A different choice for the constants c_i is usable to derive the Hawking temperature. Since the modes (U, V) and (\bar{U}, \bar{V}) are related by a Bogoliubov transformation, one has to simply determine the Bogoliubov coefficients to get the Hawking temperature [13]. By setting $\bar{c}_2 = 1$ and

$$\bar{c}_1 = c_1 \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} + c_2 \frac{\Gamma(2-c)\Gamma(a-b)}{\Gamma(a-c+1)\Gamma(1-b)} = 0, \quad (28)$$

¹Of course, this does not mean that the rotating metric is the same as the non-rotating metric because orbifoldings are different. For a related issue, see ref. [12].

c_1 and c_2 become the Bogoliubov coefficients:

$$\bar{V}_{\hat{E}\hat{N}} = c_1 U_{\hat{E}\hat{N}} + c_2 V_{\hat{E}\hat{N}}. \quad (29)$$

Thus, the expectation value of the number operator N_{EN} for (U, V) mode particles in the vacuum of the (\bar{U}, \bar{V}) mode $|\bar{0}\rangle$ is given by

$$\langle \bar{0} | N_{EN} | \bar{0} \rangle = \frac{|c_1|^2}{|c_2|^2 - |c_1|^2}. \quad (30)$$

From (28), we get

$$\langle \bar{0} | N_{EN} | \bar{0} \rangle = \frac{R}{1 - R}, \quad (31)$$

where R is the reflection coefficient in (25). In the limit $(E \pm N)/(r_+ \pm r_-) \ll \lambda$, this expression reduces to

$$\langle \bar{0} | N_{EN} | \bar{0} \rangle = \frac{1}{e^{(E - N\Omega)/T_{Hawk.}} - 1}, \quad (32)$$

with

$$T_{Hawk.} = \frac{r_+^2 - r_-^2}{2\pi r_+}, \quad \Omega = \frac{r_+}{r_-}. \quad (33)$$

Since $\Omega = J/2r_+^2$, Ω is the angular velocity of the horizon. Eq. (32) is the correct distribution function for the rotating black hole with the Hawking temperature $T_{Hawk.}$ and the angular velocity of the horizon Ω . One can regard the limit $(E \pm N)/(r_+ \pm r_-) \ll \lambda$ as the “large radial momentum” limit because λ plays the role of the radial momentum as discussed in section 3.

The dependence $E - N\Omega$ on (32) rather than E is a sign of rotating black holes. This represents the effect of the rotation and has following consequences [13].

First, note that the effect of the rotation enters into the thermal spectrum in the same way as a chemical potential. The angular momentum of the black hole plays the role of a chemical potential. This factor (32) is larger for positive N than for negative N . Thus, it is favorable to emit particles with angular momenta in the same direction as that of the black hole. As a result of the emission, the black hole rotation will be slowed down.

Also, (32) is negative when $E < N\Omega$. This has an interesting consequence. For simplicity, consider the limit $T_{Hawk.} \rightarrow 0$. For static black holes where $\Omega = 0$, the Hawking emission dies away in this limit; $\langle \bar{0}|N|\bar{0} \rangle \rightarrow 0$. However, for rotating black holes, $\langle \bar{0}|N|\bar{0} \rangle$ does not die away when $E < N\Omega$. In fact, $\langle \bar{0}|N|\bar{0} \rangle \rightarrow -1$. This negative flux means that the black hole induces stimulated emission. This phenomenon is known as super-radiance.

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